ME 690-001 **HW Assignment #1** **Spring 2017**

(Solution of Elliptic PDEs)

In this assignment you will run a computer code containing several different versions of SOR to solve problems associated with elliptic PDEs. In particular, you will run the following methods, all of which have been discussed in lectures:

i) basic "point" SOR,

ii) red-black ordered point SOR,

iii) line SOR (SLOR),

iv) red-black ordered SLOR.

In addition, you will employ a Krylov-subspace-based method, also contained in the code, called BiCGStab(ℓ). All methods are iterative, and you will begin these iterations with the initial guess $u^{(0)}(x,y) \equiv 0$.

You will first solve Dirichlet problems of the general form

$$au_{xx} + bu_{xy} + cu_{yy} = f(x,y)\,, \qquad (x,y) \in \Omega\,, \tag{1}$$

with

$$u(x,y) = g(x,y)\,, \quad \text{for } (x,y) \in \partial\Omega\,.$$

Here, $\Omega \equiv (0,1) \times (0,1)$, and the coefficients a, b, c will be assigned different constant values for various runs of the code, as described below. Then you will consider other boundary conditions.

1. Construct a centered-difference approximation to this PDE, including the mixed-derivative term. Discuss your treatment of boundary conditions. Then perform truncation error analyses of this discrete approximation to demonstrate its consistency and formal order of accuracy. (Note: details of the methods needed to accomplish this require only Taylor series expansions and can be found in the MA 537 lecture notes.)

2. With $a = c \equiv 1.0$ and $b \equiv 0.0$, set $u(x,y) \equiv 1.0$ on $\partial\Omega$ and $f(x,y) \equiv 0.0$ in the interior of Ω. Solve this problem with optimal point SOR on a grid of 101×101 points using the following max-norm iteration tolerances: 10^{-2}, 10^{-4}, 10^{-6}, 10^{-8}, 10^{-10}. Note that the exact solution to this problem is $u(x,y) \equiv 1.0$ on $\overline{\Omega}$. From the output file ellptcslvr.qqq find the maximum absolute discrepancy from this exact value, and tabulate this along with run time and number of iterations for each value of tolerance. Make contour plots (using FieldView, TecPlot, MatLab or whatever else you might prefer) for each value of tolerance; and present these, along with the table of maximum errors, in your writeup. Repeat this exercise for a grid of 501×501 points; and include the results in the previous table. Discuss the different behaviors you observe, and attempt to explain why they occur. In particular, discuss why the max-norm iteration error (shown in your output to the screen) is smaller than the max-norm solution error.

3. Use the same PDE and boundary conditions as in Prob. 2; but now require a convergence tolerance $\epsilon = 10^{-12}$, and calculate on grids of 26×26, 51×51, 101×101, 201×201, 401×401 and 801×801 points using optimal point SOR. Make a plot of required number of iterations vs. $\sqrt{N}$ where N is the total number of grid points. Discuss how your results do, or do not, compare with SOR theory. Now repeat these calculations for each of the other three forms of SOR, and for the BiCGStab method. Make plots of required number of iterations vs. $\sqrt{N}$ for each of these, and discuss to what extent these compare. Do SOR results agree with theory? Finally, plot run time vs. N for all five methods on the <u>same</u> plot, and discuss how these methods compare.

4. You will now investigate the effects of a mixed derivative on the value of the optimal SOR parameter and on behavior of iterations, in general. For the basic problem in Eq. (1) set $a = c = 1$, and use $b = 1.0, 1.5, 2.0$ in different sets of runs. Employ Dirichlet conditions in all cases with $u = 1$ on $\partial\Omega$; and set $f(x, y) \equiv 0$ in the interior of Ω. Use the same set of grids and convergence tolerance as in Prob. 3, and on each of these find the optimal iteration parameter for point SOR for each value of b. Note that in constant-coefficient cases it is possible to analytically produce a fairly good estimate of ω_b, and this is included in the code. Thus, in order to find ω_b via numerical experiments, you should first make a run with the default entry 0. for the parameter value to determine (from the code output) the nearly optimal one. Then perturb this to find the true optimal value via numerical experiments. Make a table (for each value of b) that includes number of grid points, the "theoretical" optimal parameter value (produced by the code), your numerically-determined value, and the required number of iterations for each of these. Then make a plot of required number of iterations vs. $\sqrt{N}$ for all three values of b and for both of these cases (theoretical and experimental). Repeat these calculations, and make corresponding tables and plots, for each of the other forms of SOR, and for the BiCGStab approach. Discuss how these results compare with the $b = 0$ case.

5. In this problem you will study effects of non-Dirichlet boundary conditions on the optimal relaxation parameter.

   (a) First, consider the "pure Neumann problem" which is mathematically ill posed in the sense that its solution(s) are nonunique. This causes numerical iterations to wander and be unable to converge when tight tolerances are required. Employ the same PDE as in Probs. 2 and 3 (i.e., $b = 0$), but set boundary conditions to $u_{\boldsymbol{n}} = 1.0$ on all boundaries. Here, $u_{\boldsymbol{n}}$ is the outward unit normal derivative. Use the flag value 0 when queried regarding ill posedness in the interactive problem set up. Use point SOR, and numerically determine the optimal relaxation parameter to four (4) accurate decimal places on grids consisting of 51×51, 101×101, 201×201, 401×401 points. Use these results to test whether SOR theory is valid for Neumann problems, and compare required number of iterations and optimal parameters with those of the Dirichlet case for corresponding grids.

   (b) Next repeat this problem with $f(x, y) \equiv 1.0$ and $u_{\boldsymbol{n}} = 0.0$ on all of $\partial\Omega$. You will now need to set the "ill-posedness flag" to 1; it is recommended that you choose the lower left-hand corner grid point as your Dirichlet point, and set $u = 0.0$ at this point. Carry out tests associated with SOR theory as in Part (a), and tabulate all results (including those of Part (a)). In particular, your table should include number of grid points, number of iterations, optimal parameter values and run times for both Parts (a) and (b).

   (c) Now repeat all of Parts (a) and (b) using the other three forms of SOR (R-B ordered, SLOR and R-B SLOR) and BiCGStab. Make tables of results for each of these methods as done in Part (b) for point SOR)

   (d) Finally, consider a more general, but well-posed, boundary-value problem. Continue to use the Laplace operator, now with $f(x, y) \equiv 0.0$, but with the following boundary conditions: on the left boundary $u(0, y) = 1.$; on the bottom boundary $u_y(x, 0) = 0.$; on the right boundary employ a Robin condition with $g(1, y) = 1.$ using the built-in Robin cofficient; and on the top boundary let $u_y(x, 1) = 0$. Begin with point SOR, and find the optimal parameters for the same grids employed for the earlier parts of this problem. Then repeat the runs using these same parameters for the other three forms of SOR; and also make runs on each of the grids using the BiCGStab method. Make tables similar to those of the other problem parts, and discuss and compare these results.

Provide summary discussions comparing all of these results, in particular, the behavior of the various methods studied as boundary condition type is changed..

6. In this final set of problems set $a = 10.$, $b = -36.$ and $c = 2.$ in Eq. (1), and consider both the Dirichlet problem and the more general boundary-value problem of Prob. 5. In both cases you will begin with point SOR, find the optimal parameters on the same grids employed in previous problems, and then proceed to the other four methods, repeating all calculations, and making all tables and plots.

   (a) For the pure Dirichlet problem set $f(x, y) = 0.$ on the interior of Ω, and let $u = 1.$ on $\partial\Omega$. This, again has an exact constant solution, and the main thing you will study is whether the more complicated differential operator changes optimal parameters significantly. Conduct numerical experiments in the manner of the previous problems to study this. Provide comparisons with previous simpler cases.

   (b) For this set of problems employ the rather general, but well-posed, boundary conditions of Prob. 5, and use the following right-hand-side forcing function

   $$f(x, y) = 50/\exp xy - 180 \sin 5\pi x \cos \sqrt{200\pi} y$$

   to obtain a non-trivial solution. Employ the same grids as used for previous problems to find optimal SOR parameters and execute the BiCGStab method. In this problem, the solution is somewhat complicated, so you are required to conduct grid-function convergence tests to determine appropriate values of grid spacing for an accurate solution. Thus, in addition to the results for convergence of iterations, you are required to make contour plots of the solution on various grids, and a table summarizing your grid-function convergence tests. You need do this for only one method.

Provide a summary consisting of several paragraphs in which you state what you have found in this exercise as well as what you believe still remains to be demonstrated for the methods considered in this study.