ME 690-001 *HW Assignment #2* **Spring 2017**

(Solution of Parabolic PDEs)

In this assignment you will solve a problem using four different types of methods for parabolic equations: *i*) Douglas–Rachford alternating direction implicit (ADI), *ii*) locally one dimensional (LOD), *iii*) δ-form Douglas–Gunn (D–G) and *iv*) unsplit backward Euler using point (you may wish to try red-black ordering for comparison) SOR as the solver at each time step. For each of these you will solve the following problem:

$$u_t - u_{xx} - u_{yy} = 0\,, \qquad (x,y) \in \Omega \equiv (0,1)\times(0,1)\,, \quad t \in (0,0.5]$$

with Dirichlet conditions

$$u(x,y,t) = 1\,, \qquad (x,y) \in \partial\Omega\,, \quad \forall \quad t \in [0,0.5]\,,$$

and initial conditions

$$u(x,y,0) = 1 - \sin\pi x\, \sin\pi y\,, \qquad (x,y) \in \Omega\,.$$

The exact solution to this problem is

$$u(x,y,t) = 1 - e^{-2\pi^2 t} \sin\pi x\, \sin\pi y\,.$$

For each method, you will conduct grid-function convergence tests employing the following combinations of space and time steps:

1. $k = 0.01$, $h = 0.01$ $(h_x = h_y = h)$,
2. $k = 0.005$, $h = 0.005$,
3. $k = 0.0025$, $h = 0.0025$,

In performing these calculations, set output frequencies in input to the code to permit grid-function convergence tests in max and L^2 norms at the following times: $t = 0.1, 0.2, 0.3, 0.4, 0.5$. Make tables of these results for each method. You must also store time series at the center point of the grid for each of these runs, and possibly others if you choose, and plot these (at least three) on the same plot for each separate method; that is, you will present four (4) plots of this type. Note that the code automatically produces the files containing these data. It is only required that you change their names (from fort.11) to something of your choosing so they will not be overwritten during each new run.

You will then demonstrate unconditional stability of each method via numerical experiments of your choice. You may wish to run the problem longer in time, and/or use much larger time steps. In particular, you should demonstrate that essentially arbitrarily large time steps can be used, even on fine grids. Think about what you should see in a time series produced by a stable method in contrast to that coming from an unstable one, and show that the methods of this exercise produce this behavior.

To complete this assignment, carry out the following steps.

1. Show that the above analytical solution is, in fact, the solution to the given PDE initial-boundary value problem.

2. **Douglas–Rachford**. Conduct all numerical experiments described above for this option of your code. Construct the required table and plots, and thoroughly discuss your findings. Do they agree with theory? If you discover discrepancies, attempt to explain them. Now employ the unsplit, SOR option of your code using the temporal discretization parameter $\theta = 1.0$ corresponding to backward Euler in time and, thus, Douglas–Rachford. Use only the theoretical optimal SOR parameter (and possibly also $\omega = 1.0$ for Gauss–Seidel), and adjust the convergence tolerance in order to as nearly as possible match the accuracy achieved with the time-split Douglas–Rachford method. Be sure to record run times for all runs.

3. **Locally one dimensional**. First carry out the same analyses, and provide tables and plots used above for Douglas–Rachford. It is suggested that you use $\theta = 0.5$, trapezoidal integration, but you may wish to compare results for the two integration methods. Carefully examine correspondence with theory (or lack thereof); and provide thorough discussion of your findings. Then explain why employing an unsplit method such as SOR is inconsistent with LOD methods.

4. **Douglas–Gunn**. Repeat all aspects of the Douglas–Rachford problem, now using the Douglas–Gunn method. In this case, the integration parameter value must be $\theta = 0.5$, leading to the Crank–Nicolson method before splitting. Be sure to also repeat the use of SOR, now employing $\theta = 0.5$ and again attempting to match accuracy of the Douglas–Gunn results. Finally, by means of numerical experiments, attempt to match Douglas–Gunn accuracy using the Douglas–Rachford method. In all cases, be sure to monitor and report required run times.

5. Compare CPU times and accuracy of computed results for the various methods, and discuss the reasons underlying these results, based on details of analytical forms of the methods. Taking these results, and convergence-rate results, into account, indicate your recommendation of method to use for this particular problem. Discuss (based on theory of the methods) whether you would expect your choice to change if a more complicated problem were to be solved.